

調和級数について (令和7年)

1. 調和級数は発散する.

$$1 + \frac{1}{2} + \frac{1}{3} + \dots = \infty$$

(証明)

(1) ヤコブ ベルヌーイによる証明(1689)

$$\begin{aligned} n > 1 \text{ について, } \quad \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \dots + \frac{1}{n^2} &\geq \frac{1}{n^2} + \frac{1}{n^2} + \dots + \frac{1}{n^2} \\ &= \frac{n^2 - n}{n^2} = 1 - \frac{1}{n} \end{aligned}$$

$$\therefore \frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{n^2} \geq 1$$

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{1}{k} &= 1 + \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{2^2}\right) + \left(\frac{1}{5} + \frac{1}{6} + \dots + \frac{1}{5^2}\right) + \left(\frac{1}{26} + \frac{1}{27} + \dots + \frac{1}{26^2}\right) + \dots \\ &\geq 1 + 1 + 1 + 1 + \dots \quad : \text{発散} \end{aligned}$$

2. オイラーによる証明(1748)

$$\begin{aligned} \frac{d}{dx} \log(1-x) &= -\frac{1}{1-x} \\ &= -(1+x+x^2+\dots) \end{aligned}$$

$$\therefore \log(1-x) = -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \dots\right) \dots\dots\dots (1)$$

$$\therefore x + \frac{x^2}{2} + \frac{x^3}{3} + \dots = \log \frac{1}{1-x}$$

$x = 1$  とおくと

$$1 + \frac{1}{2} + \frac{1}{3} + \dots = \infty$$

$$(1) \text{ より, } \quad \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots\dots\dots (2)$$

### 3. オイラーの定数

次の極限值は存在する. その極限值をオイラーの定数と言い,  $\gamma$ と記す.

$$\gamma = \lim_{n \rightarrow \infty} \left\{ \sum_{k=1}^n \frac{1}{k} - \log(n+1) \right\}$$

証明 (2)で  $x = \frac{1}{n}$  とおくと

$$\log \frac{n+1}{n} = \frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \frac{1}{4n^4} + \dots$$

$$\frac{1}{n} - \log(n+1) + \log n = \frac{1}{2n^2} - \frac{1}{3n^3} + \frac{1}{4n^4} - \dots$$

$$1 - \log 2 = \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \dots \dots \dots \textcircled{1}$$

$$\frac{1}{2} - \log 3 + \log 2 = \frac{1}{2 \cdot 2^2} - \frac{1}{3 \cdot 2^3} + \frac{1}{4 \cdot 2^4} - \dots \textcircled{2}$$

$$\frac{1}{3} - \log 4 + \log 3 = \frac{1}{2 \cdot 3^2} - \frac{1}{3 \cdot 3^3} + \frac{1}{4 \cdot 3^4} - \dots \textcircled{3}$$

.....

$$\frac{1}{n} - \log(n+1) + \log n = \frac{1}{2 \cdot 3^2} - \frac{1}{3 \cdot 3^3} + \frac{1}{4 \cdot n^4} - \dots \textcircled{4},$$

①+②+③+④と辺々を加え合わせると,

$$\sum_{k=1}^n \frac{1}{k} - \log(n+1) = \frac{1}{2} \sum_{k=1}^n \frac{1}{k^2} - \frac{1}{3} \sum_{k=1}^n \frac{1}{k^3} + \frac{1}{4} \sum_{k=1}^n \frac{1}{k^4} + \dots,$$

左辺を $c_n$ とおく,

$$c_n = \sum_{k=1}^n \frac{1}{k} - \log(n+1)$$

$$\begin{aligned} c_{n+1} - c_n &= \left\{ \sum_{k=1}^{n+1} \frac{1}{k} - \log(n+2) \right\} - \left\{ \sum_{k=1}^n \frac{1}{k} - \log(n+1) \right\} \\ &= \frac{1}{n+1} - \log \frac{n+2}{n+1} \end{aligned}$$

$$= \frac{1}{n+1} - \int_{n+1}^{n+2} \frac{1}{x} dx$$

$$= \int_{n+1}^{n+2} \left( \frac{1}{n+1} - \frac{1}{x} \right) dx$$

$$n+1 \leq x \leq n+2 \quad \text{で,} \quad \frac{1}{n+1} > \frac{1}{x}$$

$$\therefore c_n < c_{n+1}$$

$$\sum_{k=1}^n \frac{1}{k} = 1 + \sum_{k=2}^n \frac{1}{k} < 1 + \int_1^n \frac{1}{x} dx = 1 + \log n < 1 + \log(n+1)$$

$$c_n = \sum_{k=1}^n \frac{1}{k} - \log(n+1) < 1$$

$$\therefore c_n < 1$$

$$\therefore c_1 < c_2 < c_3 < \dots < c_n < \dots < 1$$

$$\lim_{n \rightarrow \infty} \left\{ \sum_{k=1}^n \frac{1}{k} - \log(n+1) \right\}: \text{収束}$$

#### 引用文献

オイラー入門 W.ダンハム 黒川 信重 他 訳 シュプリンガー ジャパン 2004